UNITED STATES NAVAL POSTGRADUATE SCHOOL

THESIS

RIEMANNIAN GEOMETRY AS A FIELD OVER ANOTHER GEOMETRY

by

George Henry Connor, Jr.

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George Henry Connor, Jr.
Major, United States Army
B.S., Massachusetts Institute of Technology, 1959

[redacted]

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ABSTRACT

The basic tensors of a Riemannian geometry are found in terms of tensor components by considering the geometry as a field over another arbitrary Riemannian geometry. The approach exhibits symmetries not previously noted. In particular the Riemann tensor of a geometry is found to decompose into a sum of tensors, each with the full symmetry of a Riemann tensor, and each dependent upon only one order of derivative of the metric tensor. Further work to explore the potential value of the approach to general relativity is proposed.

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I. Introduction.

The general theory of relativity of Einstein is considered as the most elegant physical theory developed to date. Its basic assumptions are merely that space-time has the geometry of a four-dimensional, normal hyperbolic Riemannian metric space, whose first fundamental form may be denoted by

$$G = g_{\mu\nu}(\chi^{\rho})\, d\chi^{\mu}\, d\chi^{\nu} \quad (0 \leq \mu, \nu, \rho \leq 3), \tag{1-1}$$

and whose metric tensor $g_{\mu\nu}$ is determined by the mass-energy contained in space-time by Einstein's field equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu} . \tag{1-2}$$

Here $R_{\mu\nu}$ is the Ricci tensor of $g_{\mu\nu}$; R is the contracted Ricci tensor; $T_{\mu\nu}$ is an "energy-momentum" tensor, specified according to the type of matter under consideration; and κ is a proportionality constant.

However, extreme difficulties arise when one tries to draw significant physical conclusions from the basic assumptions of the theory. These difficulties arise primarily due to the extreme non-linearity of the field equation. But important problems arise also due to the lack of an a priori space-time topology, due to the presence of complicated geometric objects such as the Christoffel symbols, and due to the difficulty (in fact, impossibility in general) of integrating tensors over a finite region. These difficulties have slowed progress in exploring all the richness inherent in Einstein's field equation.

For the purpose of adding to the tools with which the field equations may be explored, a new mathematical approach to the problem is proposed, and the ensuing equations and preliminary results are developed in this paper. The approach might best be described as considering one geometry as a field over another geometry, with the basic field variable

being the difference between the metric tensors of the two geometries. It is not suggested that there exists some "base" geometry (which would be in conflict with the basic concepts of Einstein's general relativity) but rather that the approach leads to relations between geometries which are of benefit in obtaining new solutions to the field equations, as well as of value in investigations of the stability of known solutions. Indeed, it would appear to add greatly to the understanding of the physical meaning of the equations.

This approach has been described as new. Certainly the metric tensor has been considered as a field over a flat (Minkowski) space before, and certainly the difference between two metric tensors has been considered as an infinitesimal in perturbation techniques before; but to the writer's knowledge, neither the generalization of these two techniques, used jointly over an arbitrary geometry and without restriction to infinitesimals, nor the results obtained by so doing have been previously reported.

II. Notation and Conventions.

Sign of definition: $\equiv$

Sign of a particular coordinate system: $\overset{*}{=}$

Einstein summation convention: $a_\mu b^\mu \equiv \sum_{u=o}^{3} a_\mu b^\mu$, etc. for repeated indices.

Signature of the metric tensor: $(+---)$; $g \equiv \det(g_{\mu\nu})$.

Choice of Riemann tensor: fixed by $V_{\mu;[\rho\sigma]} = \frac{1}{2} V_\tau R^\tau{}_{\mu\rho\sigma}$.

Choice of Ricci tensor: $R_{\mu\nu} = R^\tau{}_{\mu\tau\nu}$; trace: $R \equiv R^\tau{}_\tau$.

Square brackets for anti-symmetrization, e.g.:

$$V_{[\mu\nu]} \equiv \frac{1}{2}(V_{\mu\nu} - V_{\nu\mu});$$

$$V_{[\mu|\sigma\rho|\nu]} = \frac{1}{2}(V_{\mu\sigma\rho\nu} - V_{\nu\sigma\rho\mu}), \text{ etc.}$$

Round brackets for symmetrization, e.g.:

$$V_{(\mu\nu\rho)} \equiv \frac{1}{6}(V_{\mu\nu\rho} + V_{\rho\mu\nu} + V_{\nu\rho\mu} + V_{\mu\rho\nu} + V_{\nu\mu\rho} + V_{\rho\nu\mu}), \text{ etc.}$$

Ordinary differentiation: $V_{\mu,\nu} \equiv \partial_\nu V_\mu$.

Difference between covariant and contravariant forms of two metric tensors:

$$h_{\mu\nu} \equiv \bar{g}_{\mu\nu} - g_{\mu\nu}$$

$$k^{\mu\nu} \equiv \bar{g}^{\mu\nu} - g^{\mu\nu} .$$

Difference between affinities (Christoffel symbols of the second kind) of two metric tensors:

$$b_{\mu\nu}^{\ \rho} \equiv \bar{\Gamma}_{\mu\nu}^{\ \rho} - \Gamma_{\mu\nu}^{\ \rho} .$$

Covariant differentiation: $V_{\mu;\nu} \equiv \nabla_\nu V_\mu$ with respect to $g_{\mu\nu}$

$V_{\mu;\bar{\nu}} \equiv \bar{\nabla}_\nu V_\mu$ with respect to $\bar{g}_{\mu\nu}$.

Repeated differentiation: $V_{\mu;\nu\rho} \equiv V_{\mu;\nu;\rho}$.

A tensor defined with respect to $\bar{g}_{\mu\nu}$ is denoted by a bar over the kernal, e.g.: $\bar{R}_{\lambda\nu\rho\sigma}$.

Difference between tensors defined in some way by two different metric tensors:[1]

$$\delta R_{\lambda\nu\rho\sigma} \equiv (\delta R)_{\lambda\nu\rho\sigma} = \bar{R}_{\lambda\nu\rho\sigma} - R_{\lambda\nu\rho\sigma}$$

$$\delta h_{\lambda\nu;\rho\sigma} \equiv (\delta h)_{\lambda\nu;\rho\sigma} = h_{\lambda\nu;\bar{\rho}\bar{\sigma}} - h_{\lambda\nu;\rho\sigma}, \text{ etc.}$$

Kronecker delta:

$$\delta^{\mu}_{\ \nu} \equiv \bar{g}^{\mu\rho}\,\bar{g}_{\rho\nu} = g^{\mu\rho}g_{\rho\nu} = \begin{cases} 1 \text{ if } \mu = \nu \\ 0 \text{ otherwise} \end{cases} .$$

Units: c (speed of light) = 1, κ = 1 [which implies $\gamma = (8\pi)^{-1}$, $m = M(8\pi)^{-1}$ for Newton's constant and the radius of gravitation respectively].

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[1]Exceptions to the kernal index system of Schouten [1], which is used otherwise.

III. Mathematical Basis of the Proposed Technique.

In order to provide a basis for the proposed technique, a few basic definitions[2] of geometry are herewith presented.

A set of n real or complex values of n ordered variables is called an <u>arithmetic point</u> and the totality of these points is called an arithmetic <u>manifold</u> $\underline{U_n}$.

Consider a set M, and let the elements of M be in one-to-one correspondence with the points of a region R_o of U_n. This is called a <u>coordinate system</u> over the elements of M. The <u>transformation of coordinates</u> in M means passing to another one-to-one correspondence between these elements and the points of a region of U_n. This is also true for a subset S of M, corresponding one-to-one to a subregion R of R_o. The geometry of S depends largely upon the set of allowed transformations, B_n. If we allow the set B_n to be all transformation of points of U_n that are analytic (or of class u, i.e., up to and including the u^{th} derivative is continuous) with $R_o = R = R'$(R' being the region of U_n to which transformed), then these transformations form a group. But if we allow the set B_n of all transformation of points of U_n that are analytic (or of class U) each in a certain region, but without the condition that these regions coincide, B_n is not a group. Such a set is called a pseudo-group.

The set M, provided with the pseudo-group B_n and with all allowable coordinate systems, i.e., all systems that can be derived from

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[2]Taken mainly from Schouten [1], though greatly condensed. Trautman [2] contains perhaps a more modern exposition, which shows the relation of geometry to topology explicitly. Anderson [3] contains an easily readable though less detailed exposition.

one specified system by transformation of B_n, is called a general geometric manifold, X_n. The elements of M are called geometric points, or points of X_n. If X_n is further restricted by the introduction of the concept of parallel transport (covariant derivative) and the thereby introduced affine connection (affinity)[3] (Christoffel symbol of the second kind in Riemannian geometry) $\Gamma_{\rho\sigma}^{\ \mu}$ is defined as being a symmetric metric connection with respect to a certain real symmetric tensor $g_{\mu\nu}$ (with inverse $g^{\mu\nu}$) by means of the equation

$$\Gamma_{\rho\sigma}^{\ \mu} \equiv \frac{1}{2} g^{\mu\tau}(g_{\tau\rho,\sigma} + g_{\tau\sigma,\rho} - g_{\rho\sigma,\tau}), \qquad (2\text{-}1)$$

then the set M with all provisions is called a Riemannian geometry V_n. Also, $g_{\mu\nu}$ is called the metric tensor[4], and when its signature contains one unit element differing in sign from the others, the space is called a normal hyperbolic Riemannian space.

A geometric object field Y is a correspondence which associates with every point p of V_n (X_n in general), and every coordinate system $\{X^\mu\}$ around p, a set of N real numbers $(y_1,\ldots,y_N)$, together with a rule which determines the new set $(y_1',\ldots,y_N')$ when a coordinate transformation is made to the coordinate system $\{X^{\mu'}\}$. This rule must be given in terms of the $(y_1,\ldots,y_N)$ and the values at p of the functions and their partial derivatives which relate the coordinate systems $\{X^\mu\}$ and $\{X^{\mu'}\}$. A geometric object y is the correspondence and the rule at one point of M. (Though this is a very general definition which includes almost everything of geometrical and physical significance, there are things which are not geometric objects, notably spinors and normal derivatives.)

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[3]Schouten [1] uses the term "linear connexion."

[4]Also known as the "fundamental" tensor.

The N numbers $(y_1,\ldots,y_N)$ are called the components of y at p with respect to the coordinate system $\{X^\mu\}$.

Of special importance to physical theories are the geometric objects tensors and tensor densities[5], which are the only objects whose transformation rule is linear and homogeneous with respect to their components. Because of these properties, an equation written entirely in terms of these quantities is of the same form in all allowed coordinate systems (i.e., the one concept of covariance normally considered trivial).

An affine space E_n is an X_n in which B_n is the group of affine transformations, i.e., the group of transformations of coordinates such that the "new" coordinates $X^{\mu'}$ are given in terms of the old coordinates X^μ by the equation

$$X^{\mu'} = A^{\mu'}_{\ \mu} X^\mu + a^{\mu'} \qquad \text{Det}\,(A^{\mu'}_{\ \mu}) \neq 0,$$

where the $A^{\mu'}_{\ \mu}$ and $a^{\mu'}$ are constants. At every point p of an X_n there is a "tangent E_n," i.e., sufficiently near any point of any X_n, the affine transformations may be used in place of the transformations of the B_n of the X_n, without changing the geometry. An E_n which is also a V_n is called flat (its Riemann tensor is zero), and conversely.

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[5]By tensor density is included all objects which transform like a tensor multiplied by a power of the Jacobian of the transformation, or by the absolute value of the Jacobian, or by the Jacobian divided by its absolute value (known as a pseudo-tensor in Physics).

IV. Geometry as a Field Over Another Geometry.

Consider two V_n, characterized by their respective metric tensors $\bar{g}_{\mu\nu}$ and $g_{\mu\nu}$. Let each have regions $\bar{W}_n$ and W_n respectively which are also V_n and which map (have a one-to-one correspondence) into the same region R_o of U_n. Further let the pseudo-groups $\bar{B}_n$ and B_n associated with $\bar{W}_n$ and W_n have a non-empty union, i.e., there are elements of $\bar{B}_n$ and B_n which are common. Then this union is also a pseudo-group, D_n. Now within the limits of the coordinate transformation of D_n, the geometric objects of $\bar{W}_n$ may be related to the points of W_n in a one-to-one relationship, namely that a geometric object associated with a point $\bar{p}$ of $\bar{W}_n$ is associated with the point p of W_n which is related to the same point m of R_o as $\bar{p}$. Thus geometric object fields of $\bar{W}_n$ become geometric object fields of W_n, limited to the coordinate transformations D_n, without change of component values. But many of the essential features of a Riemannian geometry (a notable exception being its global topology) are contained in the metric tensor field, and in the Riemann tensor field, Ricci tensor field, and other fields of the tensors formed from $\bar{g}_{\mu\nu}$ and its derivatives. Therefore, the study of these fields, over any geometry possible, is essentially a study of the geometry(s) which the fields help define. It is in this sense that the expression "geometry as a field over another geometry" is used.

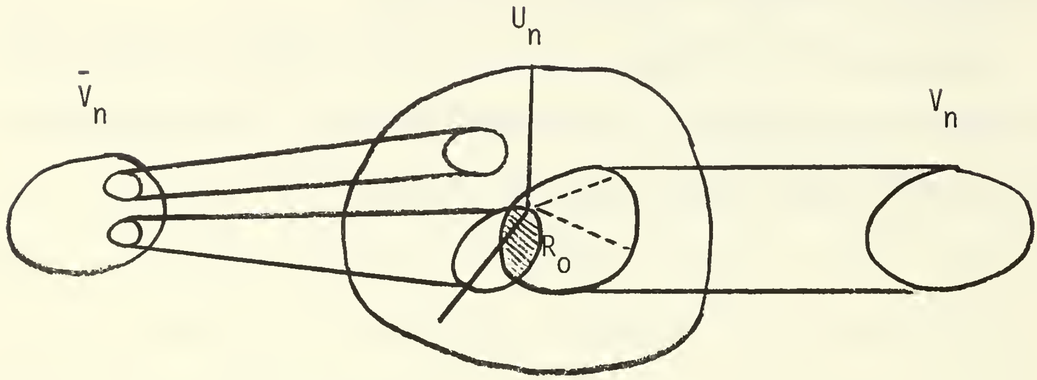
The above is perhaps made clearer by the diagram in Figure 1.

A simple two-dimensional example is that of a sphere considered as a field over a Euclidian plane. Thus, the Euclidian plane maps identically into U_2 under all coordinate transformations, whereas part of the sphere (but never all points) can be mapped one-to-one and continuously to some region (to include the totality) of U_2. Because B_n of the

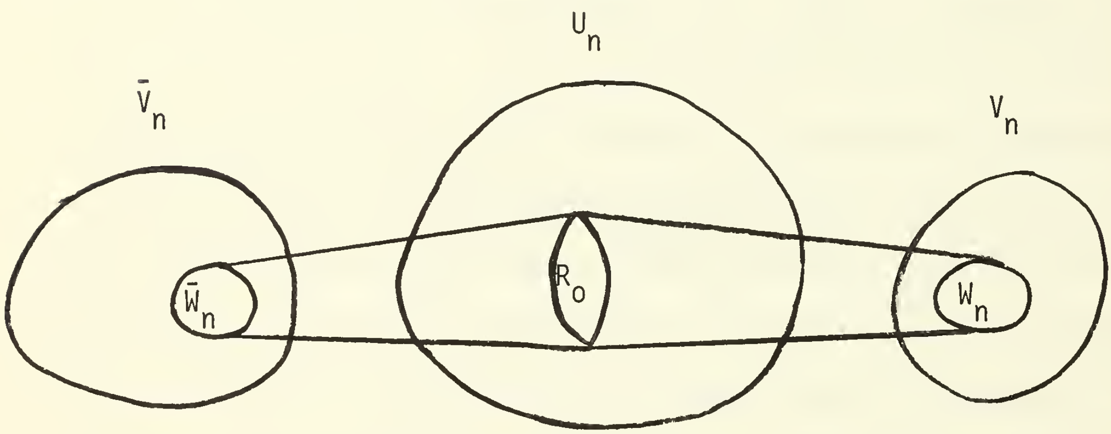
plane includes all coordinate transformations, the union D_n of B_n and $\bar{B}_n$ of the sphere is just $\bar{B}_n$. Thus, the geometric object fields defined on that part of the sphere mapped into the plane may be considered as a field over the corresponding region of the plane.

Because of the tangent E_n at every point of any X_n, any V_n may be considered as a field over a flat space at least infinitesimally (in the metric sense) from a given point of the V_n.

(a) Coordinate System Over an Entire V_n; and a $\bar{V}_n$ Such That a Single Coordinate System Will Not Cover It.



(b) $\bar{W}_n$ and W_n From (a).



(c) Relating A Geometric Object of $\bar{W}_n$ to a Point of W_n.

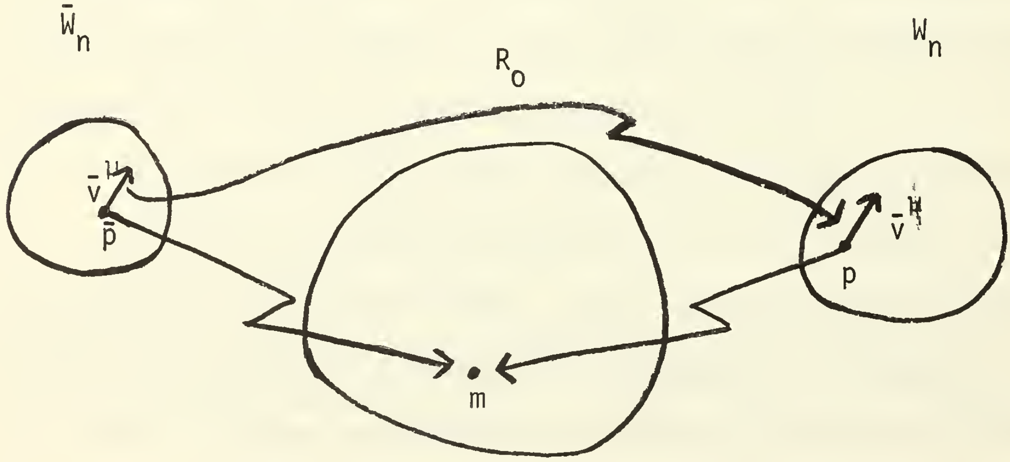


Figure 1.

V. Geometric Objects and the Physical Significance of Geometry.

The most important geometric objects in both Riemannian geometry and, therefore, general relativity, are the metric tensor $g_{\mu\nu}$; the affine connection $\Gamma_{\mu\nu}^{\rho}$ (3-1); the Ricci tensor, which in terms of $\Gamma_{\mu\nu}^{\rho}$ (and hence $g_{\mu\nu}$) is

$$R_{\mu\nu} = 2\,\Gamma^{\tau}{}_{\mu[\nu,\tau]} + 2\,\Gamma^{\times}{}_{\mu[\nu}\Gamma^{\tau}{}_{\tau]\times} \; ; \tag{5-1}$$

the Riemann tensor, where the form $R^{\tau}{}_{\mu\rho\nu}$ in terms of $g_{\mu\nu}$ is

$$R^{\tau}{}_{\mu\rho\nu} = 2\,\Gamma^{\tau}{}_{\mu[\nu,\rho]} + 2\,\Gamma^{\tau}{}_{\times[\rho}\Gamma^{\times}{}_{\nu]\mu} \; ; \tag{5-2}$$

and the Weyl (conformal curvature) tensor, which is the portion of $R_{\lambda\nu\rho\sigma}$ that is traceless with respect to $g_{\mu\nu}$,

$$C^{\tau\lambda}{}_{\rho\sigma} = R^{\tau\lambda}{}_{\rho\sigma} - 2\,\delta^{[\tau}_{[\rho}(R^{\lambda]}{}_{\sigma]} - \frac{1}{6} R\,\delta^{\nu]}{}_{\sigma]}) \; . \tag{5-3}$$

The symmetries of these objects are:

$$g_{\mu\nu} = g_{(\mu\nu)} \; ; \; \Gamma_{\mu\nu}^{\rho} = \Gamma_{(\mu\nu)}^{\rho} \; ; \; R_{\mu\nu} = R_{(\mu\nu)} \; ; \tag{5-4}$$

$$R_{\lambda\nu\rho\sigma} = R_{[\lambda\nu][\rho\sigma]} = R_{\rho\sigma\lambda\nu} \; ; \; R_{\lambda[\nu\rho\sigma]} = 0 \tag{5-5}$$

$$C_{\lambda\nu\rho\sigma} = C_{[\lambda\nu][\rho\sigma]} = C_{\rho\sigma\lambda\nu} \; ; \; C_{\lambda[\nu\rho\sigma]} = 0; \; C_{\lambda}{}^{\tau}{}_{\tau\sigma} = 0 \; . \tag{5-6}$$

There are many important identities. In particular are the Ricci identities, for example

$$T_{\rho\sigma;[\mu\nu]} = T_{\tau(\rho}R^{\tau}{}_{\sigma)\mu\nu} \; , \tag{5-7}$$

which are the conditions of integrability when covariant differentiation is used; the Bianchi identity

$$R^{\tau}{}_{\mu[\rho\nu;\sigma]} = 0 \tag{5-8}$$

and its contracted form

$$(g^{\sigma\mu}R_{\mu\nu} - \delta^{\sigma}{}_{\nu}R)_{;\sigma} = 0 \tag{5-9}$$

which is considered as a conservation equation in general relativity.

Einstein's theory of general relativity takes (1-2) as the basic field equation, thus saying that the space-time manifold (a V_n) is determined by the matter-energy contained within it. By this assumption,

gravitational forces are expressed as the geometric state. Reversing this statement, and changing to a field theoretical point of view, one could say that the geometric state, i.e., the metric tensor, is the potential field of gravitation. Thus, by analogy, $\Gamma_{\mu\nu}^{\ \ \rho}$ would be interpreted as the force field of gravitation. In fact, $\Gamma_{\mu\nu}^{\ \ \rho}$ appears in the equation of a geodesic, taken as the path of a free body when time like, as a force term, i.e.,

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma_{\nu\rho}^{\ \ \mu} \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\tau} = 0 \tag{5-10}$$

where

$d\tau = (\varepsilon\, g_{\mu\nu}\, dx^\mu\, dx^\nu)^{\frac{1}{2}}$ (ε insuring $d\tau$ is real), and may be taken as a proper time.

A further assumption of general relativity is the principle of covariance (strong form), i.e., that all coordinate systems are equivalent, or that the B_n of the V_n of space-time should include all transformations (satisfying certain differentiability conditions). Another principle, originally considered as an assumption, but which has been shown to follow from the other assumptions, is the principle of equivalence. This may be expressed by saying that locally, i.e., regions in which the variation in the gravitational field is unobservably small, a physical system linearly accelerated relative to an inertial frame [i.e., a frame (coordinate system) which is freely falling, and in which particles move rectilinearly according to special relativity (i.e., non-rotating frame)[6]] is identical to a system at rest in a gravitational field. However, if the gravitational field variation is not very small, there will be a difference. The measure of the variation is

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[6] F. A. E. Pirani [4].

the Riemann tensor, with the appropriate equation being the equation of geodesic deviation of geometry

$$\frac{D^2\eta^\mu}{D\tau^2} + R^\mu{}_{\nu\rho\sigma} V^\nu \eta^\rho V^\sigma = 0 . \tag{5-11}$$

Here D represents the absolute derivative along the geodesics of one test particle with tangent vector V^μ, and η^μ is the projection of the vector connecting two adjacent test particles onto the three space orthogonal to V^μ. The physical meaning perhaps is clearer in the diagrams of Figure 2.

A further requirement on the theory is that it reduce to Newtonian gravitational theory in the limit of very weak fields. The above equations do so.

An important phenomenon possible in the theory is that of gravitational radiation. Since such an effect would be observable in free space (i.e., no non-gravitational energy, except the test particle), the Weyl tensor is the natural quantity to consider, for it equals the Riemann tensor in free space and is not determined by the field equation. By means of its dual space symmetries (roughly speaking, the symmetries of the pairs of anti-symmetric indices), which are most clearly presented by means of spinors,[7] the Weyl tensor, and hence all free space solutions, has been classified in terms of its principle vectors. This classification is based on the symmetries of the Weyl tensor relative to a given metric tensor.

With the above as motivation, the geometric objects mentioned will be obtained as fields on another geometry, and the resulting symmetries explored.

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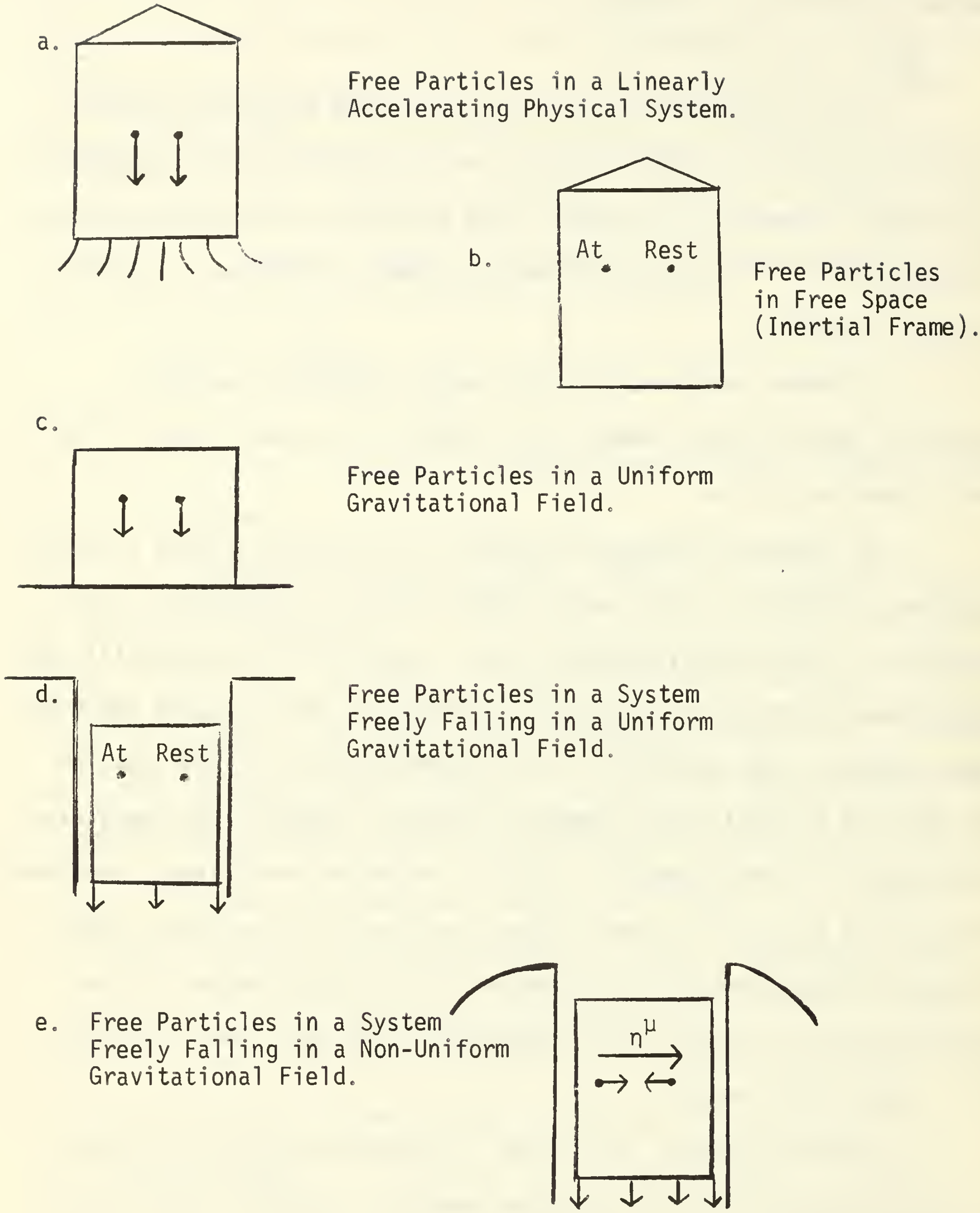
[7] Ibid.



a and c, and b and d are equivalent pairs in general relativity.

e shows the effect of a non-uniform field, as given by (5-11).

Figure 2.

VI. Derivation of Equations.

Assume that two normal hyperbolic Riemannian geometries, characterized by $\bar{g}_{\mu\nu}$ and $g_{\mu\nu}$, may be expressed in the same coordinate system over some region of each. It is desired to find the differences between tensors pertaining to the respective geometries in terms of the differences

$$h_{\mu\nu} \equiv \bar{g}_{\mu\nu} - g_{\mu\nu} , \quad k^{\mu\nu} \equiv \bar{g}^{\mu\nu} - g^{\mu\nu} , \tag{6-1}$$

which are symmetric tensors, since $\bar{g}_{\mu\nu}$ and $g_{\mu\nu}$ are symmetric. The relationship between $k^{\mu\nu}$ and $h_{\mu\nu}$ may be taken to be that resulting from the identity

$$\bar{g}^{\mu\rho} \bar{g}_{\rho\nu} = g^{\mu\rho} g_{\rho\nu}$$

to be

$$h_{\mu\rho} g^{\rho\nu} = - \bar{g}_{\mu\rho} k^{\rho\nu} . \tag{6-2}$$

There are many identities relating $h_{\mu\nu}$ to $k^{\mu\nu}$, as there are many identical ways of writing all of the equations below. A lengthly list is compiled as an appendix for reference.

The difference between the two affine connections

$$b_{\mu\nu}^{\ \ \rho} \equiv \bar{\Gamma}_{\mu\nu}^{\ \ \rho} - \Gamma_{\mu\nu}^{\ \ \rho} \tag{6-3}$$

is known to be a tensor, as can be easily seen from the transformation rule of $\Gamma_{\mu\nu}^{\ \ \rho}$,

$$\Gamma_{\mu'\nu'}^{\ \ \ \rho'} \frac{\partial x^\sigma}{\partial x^{\rho'}} = \Gamma_{\mu\nu}^{\ \ \sigma} \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} + \frac{\partial^2 x^\sigma}{\partial x^{\mu'} \partial x^{\nu'}} . \tag{6-4}$$

Furthermore, by the symmetry of $\Gamma_{\mu\nu}^{\ \ \rho}$ (5-4),

$$b_{\mu\nu}^{\ \ \rho} = b_{(\mu\nu)}^{\ \ \ \ \rho} . \tag{6-5}$$

Now consider any tensor defined on both spaces in the regions of interest. The covariant derivative of the tensor with respect to a given metric tensor is a tensor given by

$$T^{\sigma\cdots}_{\rho\cdots;\mu} = T^{\sigma\cdots}_{\rho\cdots,\mu} + \Gamma^{\sigma}_{\tau\mu} T^{\tau\cdots}_{\rho\cdots} + \cdots - \Gamma^{\tau}_{\rho\mu} T^{\sigma\cdots}_{\tau\cdots} - \cdots . \tag{6-6}$$

The difference between this tensor and the tensor defined by the covariant derivative with respect to another metric tensor is also a tensor,

$$\delta T^{\sigma\ldots}_{\rho\ldots;\mu} \equiv T^{\sigma\ldots}_{\rho\ldots;\bar{\mu}} - T^{\sigma\ldots}_{\rho\ldots;\mu} = b^{\sigma}_{\tau\mu} T^{\tau\ldots}_{\rho\ldots} + \ldots$$

$$- b^{\tau}_{\rho\mu} T^{\sigma\ldots}_{\tau\ldots} - \ldots \quad . \tag{6-7}$$

It is notable that in general $\delta T^{\sigma\ldots}_{\rho\ldots;\mu}$ equals zero only if $T^{\sigma\ldots}_{\rho\ldots}$ is a scalar T, i.e., $T_{;\mu}$ is a gradient $T_{,\mu}$. Also the other natural derivative (curl, or rot) of a covariant tensor, i.e.,

$$T_{[\sigma\ldots\rho;\mu]} = T_{[\sigma\ldots\rho,\mu]} \tag{6-8}$$

does not depend upon $\Gamma^{\rho}_{(\mu\nu)}$. Thus both gradients and curls are invariant under change of the metric tensor if the base tensor is otherwise independent of the metric tensor.

Similarly for a tensor density of weight W, whose rule of transformation is

$$\Xi^{\sigma'\ldots}_{\rho'\ldots} = \left|\frac{\partial X}{\partial X'}\right|^{W} \frac{\partial X^{\sigma'}}{\partial X^{\sigma}} \ldots \frac{\partial X^{\rho}}{\partial X^{\rho'}} \ldots \Xi^{\sigma\ldots}_{\rho\ldots} \quad , \tag{6-9}$$

(where $|\partial X/\partial X'|$ is the inverse Jacobian of the transformation), and whose covariant derivative is given by

$$\Xi^{\sigma\ldots}_{\rho\ldots;\mu} = \Xi^{\sigma\ldots}_{\rho\ldots,\mu} + \Gamma^{\sigma}_{\tau\mu} \Xi^{\tau\ldots}_{\rho\ldots} + \ldots - \Gamma^{\tau}_{\rho\mu} \Xi^{\sigma\ldots}_{\tau\ldots} \tag{6-10}$$

$$- \ldots - W\Gamma^{\tau}_{\tau\mu} \Xi^{\sigma\ldots}_{\rho\ldots} \quad .$$

Thus

$$\delta\Xi^{\sigma\ldots}_{\rho\ldots;\mu} \equiv \Xi^{\sigma\ldots}_{\rho\ldots;\bar{\mu}} - \Xi^{\sigma\ldots}_{\rho\ldots;\mu} = b^{\sigma}_{\tau\mu} \Xi^{\tau\ldots}_{\rho\ldots} + \ldots - b^{\tau}_{\rho\mu} \Xi^{\sigma\ldots}_{\tau\ldots} - \ldots$$

$$- Wb^{\tau}_{\tau\mu} \Xi^{\sigma\ldots}_{\rho\ldots} \quad , \tag{6-11}$$

where, like (6-7) is a sum of tensors, this is a sum of tensor densities, all of weight W. Note that it will be shown later that if $\bar{g}$ is proportional to g, then $b^{\tau}_{\tau\rho}$ is zero, and (6-11) is the same form as (6-7).

The invariant (upon change of $g_{\mu\nu}$) natural derivatives of tensor densities are the divergences of contravariant totally anti-symmetric tensor densities, i.e.,

$$\Xi^{[\sigma\rho\ldots\mu]}{}_{;\mu} = \Xi^{[\sigma\rho\ldots\mu]}{}_{,\mu} \qquad (6\text{-}12)$$

where the number of anti-symmetric indices is equal or less than the dimensionality of the space (for greater, such a density or tensor vanishes).

Using the fact

$$\bar{g}_{\mu\nu;\bar{\rho}} = g_{\mu\nu;\rho} = 0 \ , \qquad (6\text{-}13)$$

one quickly obtains

$$\delta\bar{g}_{\mu\nu;\rho} \equiv \bar{g}_{\mu\nu;\bar{\rho}} - \bar{g}_{\mu\nu;\rho} = -\bar{g}_{\mu\nu;\rho} = -h_{\mu\nu;\rho} \ . \qquad (6\text{-}14)$$

From this follows an equation which clearly shows the symmetry of the notation as to which geometry is considered a field over which,

$$\delta h_{\mu\nu;\rho} \equiv h_{\mu\nu;\bar{\rho}} - h_{\mu\nu;\rho} = -(g_{\mu\nu;\bar{\rho}} + \bar{g}_{\mu\nu;\rho}) \ . \qquad (6\text{-}15)$$

The term $k^{\mu\nu}{}_{;\rho}$ shall always be changed to an expression in $h_{\mu\nu;\rho}$ in this paper by the relation

$$k^{\mu\nu}{}_{;\rho} = \bar{g}^{\mu\alpha}\,\bar{g}^{\nu\beta}\,h_{\alpha\beta;\rho} \ , \qquad (6\text{-}16)$$

which follows directly from (6-2).

Using (6-7), (6-14) is

$$\delta\bar{g}_{\mu\nu;\rho} = -h_{\mu\nu;\rho} = -2\,\bar{g}_{\tau(\mu}b^{\tau}{}_{\nu)\rho}\ ^{8}. \qquad (6\text{-}17)$$

The inverse relationship of $b^{\rho}_{\mu\nu}$ in terms of $h_{\mu\nu;\rho}$ can be obtained by noting that

$$h_{\mu(\nu;\rho)} = \bar{g}_{\tau(\mu}b^{\tau}{}_{\nu)\rho} + \bar{g}_{\tau(\mu}b^{\tau}{}_{\rho)\nu} = \bar{g}_{\tau(\nu}b^{\tau}{}_{\rho)\mu} + \bar{g}_{\tau\mu}b^{\tau}{}_{\nu\rho} \ . \qquad (6\text{-}18)$$

Thus

$$h_{\mu(\nu;\rho)} - \frac{1}{2}h_{\nu\rho;\mu} = \bar{g}_{\tau\mu}b^{\tau}{}_{\nu\rho}$$

or

$$b^{\times}_{\nu\rho} = \frac{1}{2}\bar{g}^{\times\mu}(h_{\mu\nu;\rho} + h_{\mu\rho;\nu} - h_{\nu\rho;\mu}) \ . \qquad (6\text{-}19)$$

This is quite similar in form to (3-1), though deriving it directly is quite tedious.

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[8]When $\bar{g}_{\mu\nu}$ is used, it is always understood it could be broken into $g_{\mu\nu} + h_{\mu\nu}$.

The result (6-19) could also have been obtained by noting the easily verified

$$h_{\mu[\nu;\rho]} = \bar{g}_{\tau[\nu}b^{\tau}{}_{\rho]\mu} \tag{6-20}$$

and then adding the symmetric part, $\frac{1}{2} h_{\nu\rho;\mu}$ to obtain $\bar{g}_{\tau\nu}b^{\tau}{}_{\rho\mu}$.

The second derivatives follow, as

$$\delta\bar{g}_{\mu\nu;\rho\sigma} \equiv \bar{g}_{\mu\nu;\bar{\rho}\bar{\sigma}} - \bar{g}_{\mu\nu;\rho\sigma} = - h_{\mu\nu;\rho\sigma} = (\delta\bar{g}_{\mu\nu;\rho})_{;\sigma} \tag{6-21}$$

and differentiating (6-17) directly

$$= - 2\, h_{\tau(\mu|;\sigma|}b^{\tau}{}_{\nu)\rho} - 2\, \bar{g}_{\tau(\mu}b^{\tau}{}_{\nu)\rho;\sigma}$$

$$= -(2\, \bar{g}_{\times(\tau}b^{\times}{}_{\mu)\sigma}b^{\tau}{}_{\nu\rho} + 2\, \bar{g}_{\times(\tau}b^{\times}{}_{\nu)\sigma}b^{\tau}{}_{\mu\rho} + 2\, \bar{g}_{\tau(\mu}b^{\tau}{}_{\nu)\rho;\sigma}) \,.$$

Just as (6-21) is not a simple relationship between $h_{\mu\nu;\rho\sigma}$ and $b^{\rho}{}_{\mu\nu;\sigma}$, so is the inverse relation complex. It follows from (6-19) that

$$(\bar{g}_{\tau\mu}b^{\tau}{}_{\nu\rho})_{;\sigma} = h_{\tau\mu;\sigma}b^{\tau}{}_{\nu\rho} + \bar{g}_{\tau\mu}b^{\tau}{}_{\nu\rho;\sigma} = 2\, \bar{g}_{\times(\tau}b^{\times}{}_{\mu)\sigma}b^{\tau}{}_{\nu\rho} + \bar{g}_{\tau\mu}b^{\tau}{}_{\nu\rho;\sigma}$$

$$= \frac{1}{2}(h_{\mu\nu;\rho\sigma} + h_{\mu\rho;\nu\sigma} - h_{\nu\rho;\mu\sigma}) \,. \tag{6-22}$$

Having obtained the first two covariant derivations of $h_{\mu\nu}$, it is an easy matter to obtain $\delta R^{\tau}{}_{\nu\rho\sigma}$ by the following argument. From the Ricci identity (5-7),

$$h_{\mu\nu;[\rho\sigma]} = - \delta(\bar{g}_{\mu\nu;[\rho\sigma]}) = - (\bar{g}_{\mu\nu;[\bar{\rho}\bar{\sigma}]} - \bar{g}_{\mu\nu;[\rho\sigma]})$$

$$= - \quad (\bar{g}_{\tau(\mu}\bar{R}^{\tau}{}_{\nu)\rho\sigma} - \bar{g}_{\tau(\mu}R^{\tau}{}_{\nu)\rho\sigma}) = - \bar{g}_{\tau(\mu}\delta R^{\tau}{}_{\nu)\rho\sigma} \,. \tag{6-23}$$

By (6-21) after some manipulation,

$$h_{\mu\nu;[\rho\sigma]} = 2\, \bar{g}_{\tau(\mu}b^{\times}{}_{\nu)[\mu}b^{\tau}{}_{\sigma]\times} + 2\, \bar{g}_{\tau(\mu}b^{\tau}{}_{\nu)[\rho;\sigma]} \tag{6-24}$$

which implies

$$\delta R^{\tau}{}_{\nu\rho\sigma} = 2\, b^{\times}{}_{\nu[\sigma}b_{\rho]}{}^{\tau}{}_{\times} + 2\, b^{\tau}{}_{\nu[\sigma;\rho]} \,. \tag{6-25}$$

This equation can be directly verified by brute force from the defining equation of $R^{\tau}{}_{\nu\rho\sigma}$ (5-2).

Note that because of the Ricci identity (6-23) also gives

$$h_{\tau(\mu}R^{\tau}{}_{\nu)\rho\sigma} = - \bar{g}_{\tau(\mu}\delta R^{\tau}{}_{\nu)\rho\sigma} \,. \tag{6-26}$$

Now

$$\bar{R}_{\lambda\nu\rho\sigma} = \bar{g}_{\tau\lambda}\bar{R}^{\tau}{}_{\nu\rho\sigma} = (g_{\tau\lambda} + h_{\tau\lambda})R^{\tau}{}_{\nu\rho\sigma} + \bar{g}_{\tau\lambda}\delta R^{\tau}{}_{\nu\rho\sigma}$$

so that

$$\delta R_{\lambda\nu\rho\sigma} = h_{\tau\lambda}R^{\tau}{}_{\nu\rho\sigma} + \bar{g}_{\tau\lambda}\delta R^{\tau}{}_{\nu\rho\sigma}, \text{ which because of (6-26) is}$$

$$\delta R_{\lambda\nu\rho\sigma} = h_{\tau[\lambda}R^{\tau}{}_{\nu]\rho\sigma} + \bar{g}_{\tau[\lambda}\delta R^{\tau}{}_{\nu]\rho\sigma} \ . \tag{6-27}$$

$\delta R_{\lambda\nu\rho\sigma}$ must have the same symmetry as $R_{\lambda\nu\rho\sigma}$ (except for its contractions with $g^{\mu\nu}$); in particular $\delta R_{\lambda\nu\rho\sigma} = \delta R_{\rho\sigma\lambda\nu}$. Therefore

$$\begin{aligned}\delta R_{\lambda\nu\rho\sigma} &= \tfrac{1}{2}(h_{\tau[\lambda}R^{\tau}{}_{\nu]\rho\sigma} + h_{\tau[\rho}R^{\tau}{}_{\sigma]\lambda\nu} + \bar{g}_{\tau[\lambda}\delta R^{\tau}{}_{\nu]\rho\sigma} \\ &\qquad + \bar{g}_{\tau[\rho}\delta R^{\tau}{}_{\sigma]\lambda\nu}) \\ &= \tfrac{1}{2}(h_{\tau[\lambda}R^{\tau}{}_{\nu]\rho\sigma} + h_{\tau[\rho}R^{\tau}{}_{\sigma]\lambda\nu} + 2\,\bar{g}_{\tau[\lambda}b^{\tau}{}_{\nu][\sigma;\rho]} \\ &\qquad + 2\,\bar{g}_{\tau[\rho}b^{\tau}{}_{\sigma][\nu;\lambda]} + 2\,\bar{g}_{\tau[\lambda}b^{\times}{}_{\nu][\sigma}b^{\tau}{}_{\rho]\times} \\ &\qquad + 2\,\bar{g}_{\tau[\rho}b^{\times}{}_{\sigma][\nu}b^{\tau}{}_{\lambda]\times})\end{aligned} \tag{6-28}$$

and also

$$h_{\tau[\lambda}R^{\tau}{}_{\nu]\rho\sigma} - h_{\tau[\rho}R^{\tau}{}_{\sigma]\lambda\nu} = -\,(\bar{g}_{\tau[\lambda}\delta R^{\tau}{}_{\nu]\rho\sigma} - \bar{g}_{\tau[\rho}\delta R^{\tau}{}_{\sigma]\lambda\nu}) \ . \tag{6-29}$$

By the use of the symmetry $R^{\tau}{}_{[\nu\rho\sigma]} = 0$, (6-29) can be written in terms of $h_{\mu\nu;\rho\sigma}$ as can be readily though tediously verified:

$$\begin{aligned}h_{\tau[\lambda}R^{\tau}{}_{\mu]\rho\sigma} - h_{\tau[\rho}R^{\tau}{}_{\sigma]\lambda\nu} &= h_{\rho\nu;[\sigma\lambda]} + h_{\rho\lambda;[\nu\sigma]} + h_{\nu\sigma;[\lambda\rho]} \\ &\qquad + h_{\lambda\sigma;[\rho\nu]} \ .\end{aligned} \tag{6-30}$$

Using (6-22), an important identity can be obtained from (6-28). Thus

$$\begin{aligned}\delta R_{\lambda\nu\rho\sigma} &= \tfrac{1}{2}(h_{\tau[\lambda}R^{\tau}{}_{\nu]\rho\sigma} + h_{\tau[\rho}R^{\tau}{}_{\sigma]\lambda\nu}) \\ &\quad + \tfrac{1}{2}(h_{[\sigma|[\lambda;\nu]|\rho]} - h_{[\sigma|[\nu;\lambda]|\rho]}) \\ &\quad - 2\,\bar{g}_{\times(\tau}b^{\times}{}_{\lambda)[\rho}b^{\tau}{}_{\sigma]\nu} + 2\,\bar{g}_{\times(\tau}b^{\times}{}_{\nu)[\rho}b^{\tau}{}_{\sigma]\lambda} \\ &\quad + \tfrac{1}{2}(h_{[\nu|[\rho;\sigma]|\lambda]} - h_{[\nu|[\sigma;\rho]|\lambda]}) \\ &\quad - 2\,\bar{g}_{\times(\tau}b^{\times}{}_{\rho)[\lambda}b^{\tau}{}_{\nu]\sigma} + 2\,\bar{g}_{\times(\tau}b^{\times}{}_{\sigma)[\lambda}b^{\tau}{}_{\nu]\rho} \\ &\quad + \bar{g}_{\tau[\lambda}b^{\times}{}_{\nu][\sigma}b^{\tau}{}_{\rho]\times} + \bar{g}_{\tau[\rho}b^{\times}{}_{\sigma][\nu}b^{\tau}{}_{\lambda]\times} \\ &= \tfrac{1}{2}(h_{\tau[\lambda}R^{\tau}{}_{\nu]\rho\sigma} + h_{\tau[\rho}R^{\tau}{}_{\sigma]\lambda\nu}) \\ &\quad + h_{[\sigma|[\lambda;\nu]|\rho]} + h_{[\nu|[\rho;\sigma]|\lambda]} + 2\,\bar{g}_{\times\tau}b^{\times}{}_{\sigma[\lambda}b^{\tau}{}_{\nu]\rho}\end{aligned} \tag{6-31}$$

$$= \frac{1}{2}[h_{\sigma\lambda;(\nu\rho)} + h_{\rho\nu;(\lambda\sigma)} - h_{\rho\lambda;(\sigma\nu)} - h_{\nu\sigma;(\rho\lambda)}]$$

$$+ 2\,\bar{g}_{\times\tau} b_{\sigma}{}^{\tau}{}_{[\lambda} b_{\nu}{}^{\times}{}_{]\rho} + \frac{1}{2}(h_{\tau[\lambda} R^{\tau}{}_{\nu]\rho\sigma} + h_{\tau[\rho} R^{\tau}{}_{\sigma]\lambda\nu}) .$$

In this form it is apparent that the anti-symmetric portion of the second covariant derivative(i.e., $h_{\mu\nu;[\rho\sigma]}$) does not appear, and that the only non-linear term in $h_{\mu\nu}$ and its derivatives is $2\,\bar{g}_{\times\tau} b_{\sigma}{}^{\tau}{}_{[\lambda} b_{\nu}{}^{\times}{}_{]\rho}$, a function of $h_{\mu\nu}$ and its first derivatives only.

It is relatively simple to obtain $\delta R^{\tau}_{\nu\rho\sigma}$ in terms of $h_{\mu\nu}$ now. Note first that by the Ricci identity and (6-30),

$$h_{\lambda\tau} R^{\tau}_{\nu\rho\sigma} = h_{\lambda\nu;[\rho\sigma]} + \frac{1}{2}(h_{\rho\nu;[\sigma\lambda]} + h_{\rho\lambda;[\nu\sigma]} + h_{\nu\sigma;[\lambda\rho]}$$

$$+ h_{\lambda\sigma;[\rho\nu]}) + \frac{1}{2}(h_{\tau[\lambda} R^{\tau}{}_{\nu]\rho\sigma} + h_{\tau[\rho} R^{\tau}{}_{\sigma]\lambda\nu}) . \qquad (6\text{-}32)$$

Thus, also using (6-2), and (6-31),

$$\begin{aligned}
\delta R^{\tau}_{\nu\rho\sigma} &\equiv \bar{g}^{\tau\lambda}\, \bar{R}_{\lambda\nu\rho\sigma} - g^{\tau\lambda}\, R_{\lambda\nu\rho\sigma} \qquad (6\text{-}33)\\
&= \bar{g}^{\tau\lambda}\, \delta R_{\lambda\nu\rho\sigma} + k^{\tau\lambda}\, R_{\lambda\nu\rho\sigma}\\
&= \bar{g}^{\tau\lambda}(\delta R_{\lambda\nu\rho\sigma} - h_{\lambda\times} R^{\times}_{\nu\rho\sigma})\\
&= \bar{g}^{\tau\lambda}[-h_{\lambda\nu;[\rho\sigma]} + \frac{1}{2}(h_{\rho\nu;[\lambda\sigma]} + h_{\rho\nu;[\lambda\sigma]}\\
&\quad + h_{\sigma\lambda;(\nu\rho)} + h_{\lambda\sigma;[\nu\rho]} - h_{\rho\lambda;(\nu\sigma)} - h_{\rho\lambda;[\nu\sigma]}\\
&\quad - h_{\nu\sigma;(\lambda\rho)} - h_{\nu\sigma;[\lambda\rho]}) + 2\,\bar{g}_{\times\tau} b_{\sigma}{}^{\tau}{}_{[\lambda} b_{\nu}{}^{\times}{}_{]\rho}]\\
&= \bar{g}^{\tau\lambda}[-h_{\lambda\nu;[\rho\sigma]} + h_{\sigma[\lambda;\nu]\rho} - h_{\rho[\lambda;\nu]\sigma}\\
&\quad + 2\,\bar{g}_{\times\tau} b^{\tau}{}_{\sigma[\lambda} b_{\nu}{}^{\times}{}_{]\rho}]
\end{aligned}$$

In order to write $\delta R_{\nu\sigma}$ in terms of $h_{\mu\nu}$, note from (6-29) and (6-26) that

$$h_{\tau\lambda} R^{\tau}_{\nu\rho\sigma} - h_{\tau\rho} R^{\tau}_{\sigma\lambda\nu} = -\,[\bar{g}_{\tau\lambda}\delta R^{\tau}_{\nu\rho\sigma} - \bar{g}_{\tau\rho}\delta R^{\tau}_{\sigma\lambda\nu}] \qquad (6\text{-}34)$$

from which it is obvious that

$$\bar{g}^{\lambda\rho}(h_{\tau\lambda} R^{\tau}_{\nu\rho\sigma} - h_{\tau\rho} R^{\tau}_{\sigma\lambda\nu}) = 0 \qquad (6\text{-}35)$$

It is simple now to write $\delta R_{\nu\sigma}$ in terms of $b^{\rho}_{\mu\nu}$ as well as $h_{\mu\nu}$ explicitly using (6-25), (6-31), and (6-35):

$$\begin{aligned}
\delta R_{\nu\sigma} &= \delta^{\rho}{}_{\tau}\delta R^{\tau}{}_{\nu\rho\sigma} \qquad (6\text{-}36)\\
&= 2\, b^{\tau}{}_{\nu[\sigma;\tau]} + 2\, b^{\times}{}_{\nu[\sigma}b^{\tau}{}_{\tau]\times}\\
&= \bar{g}^{\rho\lambda}[\delta R_{\lambda\nu\rho\sigma} - \tfrac{1}{2}(h_{\lambda\alpha}R^{\alpha}_{\nu\rho\sigma} + h_{\rho\alpha}R^{\alpha}_{\rho\lambda\nu})]\\
&= \bar{g}^{\rho\lambda}[\tfrac{1}{2}(h_{\rho\nu;(\lambda\sigma)} + h_{\sigma\lambda;(\nu\rho)} - h_{\rho\lambda;(\nu\sigma)} - h_{\nu\sigma;(\lambda\rho)})\\
&\quad - \tfrac{1}{2}(h_{\lambda\nu;[\rho\sigma]} + h_{\rho\sigma;[\lambda\nu]}) + 2\, \bar{g}_{\times\tau}b_{\sigma}{}^{\tau}{}_{[\lambda}b_{\nu}{}^{\times}{}_{]\rho}]\\
&= \bar{g}^{\rho\lambda}[\tfrac{1}{2}(2h_{\lambda(\sigma;\nu)\rho} - h_{\rho\lambda;(\nu\sigma)} - h_{\nu\sigma;(\lambda\rho)}) + 2\, \bar{g}_{\times\tau}b_{\sigma}{}^{\tau}{}_{[\lambda}b_{\nu}{}^{\times}{}_{]\rho}] .
\end{aligned}$$

The curvature scalar follows immediately:

$$\begin{aligned}
\delta R &\equiv \bar{g}^{\mu\nu}\,\bar{R}_{\mu\nu} - g^{\mu\nu}\,R_{\mu\nu} = k^{\mu\nu}\,R_{\mu\nu} + \bar{g}^{\mu\nu}\,\delta R_{\mu\nu} \qquad (6\text{-}37)\\
&= \bar{g}^{\mu\tau}\,h_{\tau\times}\,R^{\times}{}_{\mu} + 2\,\bar{g}^{\mu\nu}\,b^{\tau}{}_{\mu[\nu;\tau]} + 2\,\bar{g}^{\mu\nu}\,b_{\mu}{}^{\times}{}_{[\nu}b^{\tau}{}_{\tau]\times}\\
&= \bar{g}^{\mu\tau}\,h_{\tau\times}\,R^{\times}{}_{\mu} + \bar{g}^{\nu\sigma}\,\bar{g}^{\lambda\rho}[h_{\rho\nu;(\lambda\sigma)} - h_{\rho\lambda;(\tau\nu)} + 2\,\bar{g}_{\times\tau}b_{\sigma}{}^{\tau}{}_{[\lambda}b_{\nu}{}^{\times}{}_{]\rho}] .
\end{aligned}$$

The other forms of the Riemann tensor, which are of importance in connection with dual space symmetries are straightforward, though it should be noted that one must be careful with the meaning of the notation. For example,

$$\delta R_{\lambda}{}^{\tau}{}_{\rho\sigma} = -\,\delta R^{\tau}{}_{\lambda\rho\sigma} \tag{6-38}$$

but

$$\delta R_{\nu}{}^{\tau}{}_{\rho\sigma} \neq g_{\nu\alpha}\, g^{\tau\beta}\, \delta R^{\alpha}{}_{\beta\nu\rho}\,,$$

and

$$\delta R_{\nu}{}^{\tau}{}_{\rho\sigma} \neq \bar{g}_{\nu\alpha}\, \bar{g}^{\tau\beta}\, \delta R^{\alpha}{}_{\beta\nu\rho}\,.$$

Now

$$\begin{aligned}\delta R^{\tau\lambda}{}_{\rho\sigma} &= \bar{g}^{\tau\mu}\, \bar{g}^{\lambda\nu}\, \bar{R}_{\mu\nu\rho\sigma} - g^{\tau\mu}\, g^{\lambda\nu}\, R_{\mu\nu\rho\sigma} \\ &= \bar{g}^{\tau\mu}\, \bar{g}^{\lambda\nu}\, \delta R_{\mu\nu\rho\sigma} + k^{\mu[\tau}g^{\lambda]\nu}\, R_{\mu\nu\rho\sigma} \\ &\qquad + k^{\tau\mu}\, k^{\lambda\nu}\, R_{\mu\nu\rho\sigma}\,,\end{aligned} \tag{6-39}$$

and by tedious manipulations

$$\begin{aligned}&= \bar{g}^{\lambda\nu}\, \bar{g}^{\mu\tau}(h_{\sigma[\mu;\nu]\rho} - h_{\rho[\mu;\nu]\sigma} \\ &\qquad\qquad + 2\, \bar{g}_{\times\alpha}\, b^{\alpha}{}_{\sigma[\mu} b_{\nu]}{}^{\times}{}_{\rho}) \\ &\quad - g^{\lambda\nu}\, g^{\mu\tau}\, h_{\beta[\mu} R^{\beta}{}_{\nu]\rho\sigma} \\ &\quad - \frac{1}{2}\, \bar{g}^{\mu\alpha}\, g^{\gamma\tau}\, g^{\lambda\nu}\, h_{\beta\mu}\, h_{\alpha[\gamma} R^{\beta}{}_{\nu]\rho\sigma}\,.\end{aligned}$$

The Weyl tensor follows immediately in the form

$$\begin{aligned}\delta C^{\tau\lambda}{}_{\rho\sigma} &= \delta R^{\tau\lambda}{}_{\rho\sigma} - 2\delta^{[\tau}_{[\rho}(\delta R^{\lambda]}{}_{\sigma]} - \frac{1}{6}\,\delta R\, \delta^{\lambda]}{}_{\sigma]}) \\ &= \delta R^{\tau\lambda}{}_{\rho\sigma} - 2\delta^{\beta}{}_{\alpha}\delta^{[\tau}_{[\rho}(\delta R^{|\alpha|\lambda]}{}_{|\beta|\sigma]} - \frac{1}{6}\,\delta^{|\varepsilon}_{|\gamma}\delta R^{\alpha\gamma|}{}_{\beta\varepsilon|}\, \delta^{\lambda]}{}_{\sigma]})\,.\end{aligned} \tag{6-40}$$

This completes the development of equations for use in this paper, and for use as a basic reference in using the technique presented. Obviously, other tensors of a particular geometry can be put into forms similar to those above.

VII. Observations Upon the Equations.

By the use of tensor notation for the derivatives of $\bar{g}_{\mu\nu}$ in the tensors of interest to general relativity, it is much easier to determine the symmetries involved, particularly those of the components of the equation. Hopefully, these symmetries will aid in determining the physical meaning of these components.

By examination of the equations obtained in the last section, it is evident that the differences between geometric tensors $R_{\lambda\nu\rho\sigma}$, $R^{\tau}{}_{\nu\rho\sigma}$, $R_{\nu\sigma}$, R, etc., are all functions of $(h_{[\sigma|[\lambda;\nu]|\rho]} + h_{[\nu|[\rho;\sigma]|\lambda]})$, $\bar{g}_{\times\tau} b_{\sigma}{}^{\tau}{}_{[\lambda} b_{\nu}{}^{\times}{}_{]\rho}$, and functions of $h_{\mu\nu}$ and the geometric tensors of the base geometry.

First consider the only portion of $h_{\sigma\lambda;\nu\rho}$ which appears in $\delta R_{\lambda\nu\rho\sigma}$, (6-31),

$$H_{\lambda\nu\rho\sigma} \equiv h_{[\sigma|[\lambda;\nu]|\rho]} + h_{[\nu|[\rho;\sigma]|\lambda]} \tag{7-1}$$

$$= h_{\sigma\lambda;(\nu\rho)} + h_{\rho\nu;(\lambda\sigma)} - h_{\rho\lambda;(\sigma\nu)} - h_{\nu\sigma;(\rho\lambda)} \cdot$$

This tensor obviously has the symmetries

$$H_{\lambda\nu\rho\sigma} = H_{[\lambda\nu][\rho\sigma]} = H_{\rho\sigma\lambda\nu}, \tag{7-2}$$

and it will be shown that indeed it has the full symmetry of a Riemannian tensor, i.e.,

$$H_{\lambda[\nu\rho\sigma]} = 0. \tag{7-3}$$

The proof is simply to note that the completely anti-symmetric portion of a tensor symmetric in two indices is zero, i.e.,

$$A_{\nu\rho\sigma} = A_{(\nu\rho)\sigma} \text{ implies } A_{[\nu\rho\sigma]} = 0. \tag{7-4}$$

But holding λ constant, it is apparent all the tensors $h_{\sigma\lambda;(\nu\rho)}$ of equation (7-1) are of this form in the indices $_{\nu\rho\sigma}$, thus proving (7-3).

Now $\bar{g}_{\times\tau} b^{\tau}{}_{\sigma[\lambda} b_{\nu]}{}^{\times}{}_{\rho}$ is also a Riemannian tensor with indices in order $\lambda\nu\rho\sigma$, for it is anti-symmetric in the $\lambda\nu$ and $\rho\sigma$ pairs, symmetric upon interchange of the pairs, and finally, since $b^{\tau}{}_{\sigma\lambda} = b^{\tau}{}_{(\sigma\lambda)}$, it also has the symmetry of (7-3) because of (7-4). But therefore, since $\delta R_{\lambda\nu\rho\sigma}$ is a Riemann tensor, so is $h_{\tau[\lambda} R^{\tau}{}_{\nu]\rho\sigma} + h_{\tau[\rho} R^{\tau}{}_{\sigma]\lambda\nu}$ (which can be proven directly, using $R^{\tau}{}_{[\nu\rho\sigma]} = 0$). Thus is proven the result that $\delta R_{\lambda\nu\rho\sigma}$ is made up of the sum of three Riemann tensors, one being a combination of components of $h_{\rho\lambda;(\nu\rho)}$, one being a non-linear function of $h_{\mu\nu}$ and its first derivatives, and one being the appropriate combination of components of $h_{\tau\lambda} R^{\tau}{}_{\nu\rho\sigma}$.

Further, the function $\bar{g}_{\times\tau} b^{\tau}{}_{\sigma[\lambda} b_{\nu]}{}^{\times}{}_{\rho}$ can be broken into a sum of Riemann tensors. For, after a tedious manipulation of identities,

$$\bar{g}_{\times\tau} b^{\tau}{}_{\sigma[\lambda} b^{\times}{}_{\nu]\rho} = \frac{\bar{g}^{\times\tau}}{4} \{ [(h_{\sigma[\lambda;|\times|} h_{\nu]\rho;\tau})] \qquad \text{(I) (7-5)}$$

$$+[(h_{\rho[\lambda;|\times} h_{\sigma\tau;|\nu]} - h_{\sigma[\lambda;|\times} b_{\rho\tau;|\nu]} \qquad \text{(II)}$$

$$+ h_{\rho[\lambda;|\times|} h_{\nu]\tau;\sigma} - h_{\sigma[\lambda;|\times|} h_{\nu]\tau;\rho})]$$

$$+[(h_{\sigma\times;[\lambda} h_{\nu]\tau;\rho} - h_{\rho\times;[\lambda} h_{\nu]\tau;\sigma}) \qquad \text{(III)}$$

$$+ (h_{\sigma\times;[\lambda} h_{|\rho\tau|;\nu]}$$

$$+ h_{\times[\lambda;|\sigma|} h_{\nu]\tau;\rho})] \},$$

where the function on each line is anti-symmetric in $\lambda\nu$ and $\rho\sigma$, the function in ( ) is symmetric in the pairs $\lambda\nu$ and $\rho\sigma$, and the functions in [ ] have the full symmetry of a Riemann tensor $R_{\lambda\nu\rho\sigma}$, i.e., $R_{\lambda[\nu\rho\sigma]} = 0$. The proofs are tedious and shall not be given. All, however, are straightforward, and the simplest method to the writer's knowledge is, after establishing that they are of the form $R_{[\lambda\nu][\rho\sigma]} = R_{\rho\sigma\lambda\nu}$, which is almost obvious, to find combinations with symmetric

pairs so that (7-4) can be used to show the $R_{\lambda[\nu\rho\sigma]} = 0$ symmetry is present. Note that though all of these Riemann tensors could be decomposed into a part contracted by $g^{\chi\tau}$ and a part contracted by $k^{\chi\tau}$, it would seem advisable to consider them in terms of $\bar{g}^{\chi\tau}$, since it appears that the physical significance lies in the contraction with $\bar{g}^{\mu\nu}$.

Before proceeding with further examination of the equation, the significance of the decomposition of the Riemann tensor into a sum of Riemannian tensors should be pointed out. First it is significant because every tensor with the Riemann tensor symmetry is a Riemann tensor of a Riemann geometry. Thus we have shown that, in this sense, each geometry is a composite of several geometries. The full meaning of this is not clear. That is, it could result that some of these "component" geometries are subspace geometries, or at least geometries of spaces with less than the dimension of the "complete" geometry. But this is certainly not true by hypothesis for the space used as a base on the preceding work, and hence for its Riemann tensor. But in any case, it is clear that each of the "component" Riemannian tensors is a function of only one order of derivatives. Thus, the second derivative of $h_{\mu\nu}$ determines one, the first derivative of $h_{\mu\nu}$ determines several, while $h_{\mu\nu}$ "interacting" with the base geometry provides another, and finally the base geometry provides the last. It is apparent that the most significant tensors are those formed by the derivatives of $h_{\mu\nu}$, for any geometry can be projected at least at a point onto its tangent affine space, which can be considered as a flat Riemannian geometry. Thus, using an orthonormal coordinate system, the "component" tensors are merely expressions of the first and second

normal derivatives, i.e., the usual equation for the Riemannian tensor. But now, the usual equation has been separated into parts which separately form Riemannian tensors, when the normal derivatives are considered as covariant derivatives in a special coordinate system.

The significance, physically and mathematically, of a Riemannian tensor has only recently been, at least partially, found. The physical significance as shown by Pirani has already been partially sketched (Chapter V). The mathematical significance in classifying the vacuum space solution by the Weyl tensor, which has physical significance in particular in regard to gravitational radiation has also been mentioned. Thus, by breaking down the Riemannian tensor into component Riemannian tensors (and hence the Weyl tensor into component Weyl tensors), particularly where these components are functions of only one derivative of the metric tensor, one is much closer to finding the physical and mathematical meaning of the metric tensor, and thus, that of the theory and equations themselves.

After that digression, let us proceed to make a preliminary examination of each of the tensors making up the Riemann tensor, and hence the other geometric tensors. It should be noted that though these tensors have the symmetry of a Riemann tensor, and hence are Riemann tensors, it is not "their" Ricci tensor nor curvature scalar which appears in the Ricci tensor and curvature scalar under consideration. Instead, it is the analogous contraction by $\bar{g}_{\mu\nu}$, rather than by the metric tensor which "belongs" to the Riemann tensor. It is these contractions that shall be obtained.

As an aid in examining the tensors, a few additional relations can be easily obtained. From the well known fact

$$\Gamma^{\tau}_{\tau\nu} = \frac{1}{2}[\log(-g)]_{,\nu} \qquad (7\text{-}6)$$

follows

$$b^{\tau}{}_{\tau\nu} = \frac{1}{2}[\log(\frac{\bar{g}}{g})]_{,\nu}. \tag{7-7}$$

But $\bar{g}/g$ is a scalar for g is a scalar density of weight one. Thus (7-7) is the gradient of a scalar, and the ordinary derivative symbol may be replaced by the covariant derivative symbol. Thus, by the theorem that a natural derivative of a natural derivative vanishes (i.e., curl of a gradient in this case),

$$b^{\tau}{}_{\tau\nu;\sigma} = b^{\tau}{}_{\tau(\nu;\sigma)} \cdot \tag{7-8}$$

From (6-19)

$$b^{\tau}{}_{\tau\nu} = \frac{1}{2}\,\bar{g}^{\tau\rho}h_{\tau\rho;\nu} \tag{7-9}$$

so

$$b^{\tau}{}_{\tau\nu;\sigma} = \frac{1}{2}\,\bar{g}^{\tau\rho}h_{\tau\rho;\nu\sigma} + \frac{1}{2}\,k^{\tau\rho}{}_{;\sigma}h_{\tau\rho;\nu} \cdot \tag{7-10}$$

But by (6-16)

$$\begin{aligned} k^{\tau\rho}{}_{;\sigma}h_{\tau\rho;\nu} &= -\,\bar{g}^{\tau\alpha}\,\bar{g}^{\rho\beta}\,h_{\alpha\beta;\sigma}h_{\tau\rho;\nu} \\ &= h_{\alpha\beta;\sigma}k^{\alpha\beta}{}_{,\nu}\ , \end{aligned} \tag{7-11}$$

i.e., $k^{\tau\rho}{}_{;\sigma}h_{\tau\rho;\nu}$ is symmetric in σ and ν. Thus

$$b^{\tau}{}_{\tau[\nu;\sigma]} = \frac{1}{2}\,\bar{g}^{\tau\rho}h_{\tau\rho;[\nu\sigma]} = 0 \tag{7-12}$$

and the trace of $h_{\tau\rho;\nu\sigma}$ with respect to $\bar{g}^{\tau\rho}$ is known in terms of the second derivatives of the ratio of determinants of the metric tensors and first derivatives of $h_{\mu\nu}$.

Thus the various tensors and their contractions may be written as given in Table 1.

In order to further explore the symmetries of these tensors (of Table 1) it is best to transform to spinor notation. This will not be done in this paper. It is worth noting that the reason that spinor notation seems to be promising for further work is the fact that one has a collection of Riemann tensors, whose properties are particularly suited for investigation by spinors.

As a conclusion to this preliminary examination of the geometric tensors, a few special cases should be pointed out in order to indicate the potential value of the point of view presented.

If the base geometry ($g_{\mu\nu}$) is taken to be that of flat space, then $R_{\lambda\nu\rho\sigma} = R_{\nu\sigma} = R = 0$, and all of the differences given in Chapter VI are actually the complete tensors of interest. The advantages of the notation per se is merely of presenting the equations in terms of covariant differentiation, rather than of ordinary differentiation. The only true advantage lies in the symmetries which are made more evident by this notation. If in addition to using a flat base geometry, one considers infinitesimal $h_{\mu\nu}$, then the equations reduce to $\bar{R}_{\lambda\nu\rho\sigma} = H_{\lambda\nu\rho\sigma}$ (7-6) and (assuming one is not near a singular point where $k^{\mu\nu}$ approaches infinity even for small $h_{\mu\nu}$) to the contractions of $H_{\lambda\nu\rho\sigma}$ by $g^{\mu\nu}$, the flat metric tensor. For orthonormal coordinates, these are identical to the usual linear approximation equations. The advantage of the notation in this respect (other than clearly showing what terms are associated with what order of approximation) lies in linear approximation from <u>other</u> than flat space.

Indeed, one need not speak only of approximation. For consider a gravitational radiation situation. At the surface of the wave, both $g_{\mu\nu}$ and $g_{\mu\nu,\sigma}$ (and hence $g_{\mu\nu;\sigma}$) are continuous, and only $g_{\mu\nu,\sigma\rho}$ (and hence $g_{\mu\nu;\sigma\rho}$) is discontinuous. Thus if one considers the space on one "side" (either "side") of the wave front as the base space, and consider $h_{\mu\nu}$ to be that change in the metric tensor caused by the discontinuous second derivative, then at the interface

$$\delta R_{\lambda\nu\rho\sigma} = H_{\lambda\nu\rho\sigma} + M_{\lambda\nu\rho\sigma;} \bar{g}^{\mu\nu} = g^{\mu\nu}. \qquad (7\text{-}13)$$

(See Table 1 for definition of $M_{\lambda\nu\rho\sigma}$).

Thus one can investigate the allowable changes in the Riemann tensor due to gravitational radiation. In particular one can work in the free space case, where at the interface the change in Riemann tensor is equal to the change in Weyl tensor, and the natural spinor space of both metrics is the same. Using this technique might lead to a general free space solution of the field equation, which has not yet been obtained.

Special cases may also turn out to be the best way of resolving more of the physical meaning of the components of the geometric tensors. For example, if $\bar{g}$ is a constant multiple of g, then many terms vanish in the equation. This occurs in the Schwarzschild solution for one case. But the general significance is not clear. Also, is it possible to have one or more of the Riemann tensors dependent only upon the first derivative vanish, without all vanishing? If so, with what significance?

The conclusion of the examination has raised more questions than it has answered. But this in itself is possibly the chief advantage in the new way of viewing the general theory of relativity.

Table 1.

Tensors of $\bar{R}_{\lambda\nu\rho\sigma}$ and Their Contractions.

$$\bar{R}_{\lambda\nu\rho\sigma} = R_{\lambda\nu\rho\sigma} + H_{\lambda\nu\rho\sigma} + I_{\lambda\nu\rho\sigma} + II_{\lambda\nu\rho\sigma} + III_{\lambda\nu\rho\sigma} + M_{\lambda\nu\rho\sigma}$$

A. $H_{\lambda\nu\rho\sigma}$

$$H_{\lambda\nu\rho\sigma} \equiv \frac{1}{2}(H_{\sigma\lambda;(\nu\rho)} + h_{\rho\nu;(\lambda\sigma)} - h_{\rho\lambda;(\sigma\nu)} - h_{\nu\sigma;(\rho\lambda)})$$

$$H_{\nu\sigma} \equiv \bar{g}^{\lambda\rho}H_{\lambda\nu\rho\sigma} = \frac{1}{2}\bar{g}^{\lambda\rho}[h_{\lambda(\sigma;\nu)\rho}+h_{\lambda(\sigma;|\rho|\nu)} - h_{\rho\lambda;(\sigma\nu)} - h_{\nu\sigma;(\rho\lambda)}]$$

$$= \frac{1}{2}\bar{g}^{\lambda\rho}[h_{\lambda(\sigma;\nu)\rho} + h_{\lambda(\sigma;|\rho|\nu)} - h_{\nu\sigma;(\rho\lambda)}]$$

$$- \frac{1}{2}(\log\frac{\bar{g}}{g});\nu\sigma - \bar{g}^{\lambda\rho}\bar{g}^{\mu\tau}h_{\lambda\mu;\sigma}h_{\rho\tau;\nu}$$

$$H \equiv \bar{g}^{\nu\sigma}H_{\nu\sigma} = \bar{g}^{\nu\sigma}\bar{g}^{\lambda\rho}(h_{\sigma\lambda;(\nu\rho)} - h_{\rho\lambda;(\sigma\nu)})$$

$$= \bar{g}^{\nu\sigma}\bar{g}^{\lambda\rho}h_{\sigma\lambda;(\nu\rho)} - \bar{g}^{\nu\sigma}(\log\frac{\bar{g}}{g});\nu\sigma - 2\bar{g}^{\nu\sigma}\bar{g}^{\lambda\rho}\bar{g}^{\mu\tau}h_{\lambda\mu;\sigma}h_{\rho\tau;\nu}$$

B. $I_{\lambda\nu\rho\sigma}$

$$I_{\lambda\nu\rho\sigma} \equiv \frac{\bar{g}^{\times\tau}}{8}(h_{\sigma\lambda;\times}h_{\nu\rho;\tau} - h_{\sigma\nu;\times}h_{\lambda\rho;\tau})$$

$$I_{\nu\sigma} \equiv \bar{g}^{\lambda\rho}I_{\lambda\nu\rho\sigma} = \frac{\bar{g}^{\lambda\rho}}{8}\bar{g}^{\times\tau}(h_{\sigma\lambda;\times}h_{\nu\rho;\tau} - h_{\sigma\nu;\times}h_{\lambda\rho;\tau})$$

$$= \frac{1}{8}\bar{g}^{\times\tau}(\bar{g}^{\lambda\rho}h_{\sigma\lambda;\times}h_{\nu\rho;\tau} - h_{\sigma\nu;\times}(\log\frac{\bar{g}}{g});\tau)$$

$$I \equiv \bar{g}^{\mu\sigma}I_{\nu\sigma} = \frac{1}{8}\bar{g}^{\times\tau}\bar{g}^{\nu\sigma}\bar{g}^{\lambda\rho}h_{\sigma\lambda;\times}h_{\nu\rho;\tau} - \frac{1}{8}\bar{g}^{\times\tau}(\log\frac{\bar{g}}{g});\times(\log\frac{\bar{g}}{g});\tau$$

C. $II_{\lambda\nu\rho\sigma}$

$$II_{\lambda\nu\rho\sigma} \equiv \frac{1}{4}\bar{g}^{\times\tau}(h_{\rho[\lambda;|\times}h_{\sigma\tau;|\nu]} - h_{\sigma[\lambda;|\times}h_{\rho\tau;|\nu]}$$

$$+ h_{\rho[\lambda;|\times|}h_{\nu]\tau;\sigma} - h_{\sigma[\lambda;|\times|}h_{\nu]\tau;\rho})$$

$$II_{\nu\sigma} \equiv \bar{g}^{\lambda\rho}II_{\lambda\nu\rho\sigma} = \frac{1}{4}\bar{g}^{\lambda\rho}\bar{g}^{\times\tau}(h_{\rho\lambda;\times}h_{\tau(\sigma;\nu)} - h_{\rho(\nu;|\times|}h_{\sigma)\tau;\lambda}$$

$$- h_{\lambda(\sigma;|\times}h_{\rho\tau|;\nu)} + h_{\sigma\nu;\times}h_{\rho\tau;\lambda})$$

$$= \frac{1}{4}\bar{g}^{\lambda\rho}\bar{g}^{\times\tau}(h_{\lambda(\sigma;|\times}h_{\rho\tau|;\nu)} - h_{\rho(\nu;|\times|}h_{\sigma)\tau;\lambda}$$

$$- h_{\lambda(\sigma;|\times}h_{\rho\tau|;\nu)}) + \frac{1}{4}\bar{g}^{\times\tau}h_{\tau(\sigma;\nu)}(\log\frac{\bar{g}}{g});\times$$

$$II \equiv \bar{g}^{\nu\sigma}II_{\nu\sigma} = \frac{1}{2}\bar{g}^{\times\tau}(\bar{g}^{\nu\sigma}h_{\tau(\nu;\sigma)}(\log\frac{\bar{g}}{g});\times - \bar{g}^{\nu\sigma}\bar{g}^{\lambda\rho}h_{\rho\nu;\times}h_{\sigma\tau;\lambda})$$

D. $III_{\lambda\nu\rho\sigma}$

$$III_{\lambda\nu\rho\sigma} \equiv \frac{1}{4}\bar{g}^{\times\tau}(h_{\sigma\times;[\lambda}h_{\nu]\tau;\rho} - h_{\rho\times;[\lambda}h_{\nu]\tau;\sigma}$$

$$+ h_{\sigma\times;[\lambda}h_{|\rho\tau|;\nu]} + h_{\times[\lambda;|\sigma}h_{\nu]\tau;\rho})$$

$$III_{\nu\sigma} \equiv \bar{g}^{\lambda\rho}III_{\lambda\nu\rho\sigma} = \frac{1}{8}\bar{g}^{\lambda\rho}\bar{g}^{\times\tau}(h_{\sigma\times;\lambda}h_{\nu\tau;\rho} + h_{\rho\times;\nu}h_{\lambda\tau;\sigma}$$

$$- 4h_{\times(\nu;\sigma)}h_{\tau(\rho;\lambda)} + 2h_{\times\lambda;(\sigma}h_{\nu)\tau;\rho})$$

$$III \equiv \bar{g}^{\nu\sigma}III_{\nu\sigma} = \frac{1}{2}\bar{g}^{\nu\sigma}\bar{g}^{\lambda\rho}\bar{g}^{\times\tau}(h_{\times(\sigma;\lambda)}h_{\tau\nu;\rho} - h_{\times(\nu;\sigma)}h_{\tau(\rho;\lambda)})$$

E. $R_{\lambda\nu\rho\sigma} + M_{\lambda\nu\rho\sigma} \equiv R_{\lambda\nu\rho\sigma} + \frac{1}{2}(h_{\tau[\lambda}R^{\tau}{}_{\nu]\rho\sigma} + h_{\tau[\rho}R^{\tau}{}_{\sigma]\lambda\nu})$

$$R_{\nu\sigma} + M_{\nu\sigma} \equiv \bar{g}^{\lambda\rho}(R_{\lambda\nu\rho\sigma} + M_{\lambda\mu\rho\sigma}) = R_{\nu\sigma} - k^{\lambda\rho}R_{\lambda\nu\rho\sigma}$$

$$+ \frac{1}{2}\bar{g}^{\lambda\rho}h_{\tau\lambda}R^{\tau}{}_{(\nu|\rho|\sigma)}$$

$$- \bar{g}^{\lambda\rho}h_{\tau(\sigma}R^{\tau}{}_{|\rho\lambda|\nu)}$$

$$= R_{\nu\sigma} - \frac{\bar{g}^{\lambda\rho}}{2}(h_{\tau(\nu}R^{\tau}{}_{\lambda)\rho\sigma} + h_{\tau(\rho}R^{\tau}{}_{\sigma)\lambda\nu})$$

$$R + M \equiv \bar{g}^{\nu\sigma}(R_{\nu\sigma} + M_{\nu\sigma}) = R + k^{\nu\sigma}R_{\nu\sigma}$$

$$= R - \bar{g}^{\nu\sigma}h_{\sigma\tau}R^{\tau}{}_{\nu}$$

VIII. Summary.

The concept of a geometry as a field over another field has resulted in placing the equations of general relativity in terms of tensor components. This has clarified some of the symmetries of the equations. In particular, it has led to the significant discovery that the Riemann tensor may be decomposed into a sum of tensors with the same symmetry, and, most interestingly, each of these tensors is dependent upon only one order of derivative of the metric tensor.

The full mathematical and physical significance of the symmetries are not found. However, in terms of the concept employed, one can say that the Riemann tensor considered as a field is composed of two tensors which depend only on the basic field variable and its covariant derivatives and which tentatively appear to contain the primary physical meaning of the field, plus an "interaction" term, a product linear in the field variable and linear in the base Riemann tensor, and finally the Riemann tensor of the base space itself. The last two mentioned tensors appeared to be treated most naturally as a sum, for upon contraction with $\bar{g}^{\mu\nu}$, the contravariant field tensor, significant components of the two cancel one another. However contraction does not mix the tensors dependent upon the derivatives of the field tensor with those independent of the derivatives of the field tensor.

A great number of avenues of approach to various problems are opened by the concept and its formulation. However, the most significant and necessary next step would appear to be the full exploration of the meaning of the various tensors with Riemann tensor symmetry.

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APPENDIX

Summary of Identities

A. Relations between $h_{\rho\nu}$, $k_{\rho\nu}$, and their derivatives.

1. $\bar{g}_{\mu\nu} - g_{\mu\nu} \equiv h_{\mu\nu}$; $\bar{g}^{\mu\nu} - g^{\mu\nu} \equiv k^{\mu\nu}$

2. (a) $h_{\rho\sigma}g^{\sigma\mu} + g_{\rho\sigma}k^{\sigma\mu} + h_{\rho\sigma}k^{\sigma\mu} = 0$

   (b) $\bar{g}^{\sigma\mu}h_{\rho\sigma} = - g_{\rho\sigma}k^{\sigma\mu}$; $g^{\rho\nu}h_{\rho\mu} = - \bar{g}_{\rho\mu}k^{\rho\nu}$

   (c) $h_{\rho\nu} = - \bar{g}_{\mu\nu}g_{\rho\sigma}k^{\sigma\mu}$; $k^{\mu\nu} = - \bar{g}^{\mu\alpha}g^{\nu\beta}h_{\alpha\beta}$

3. $h_{\rho\nu;\times} = \bar{g}_{\rho\nu;\times}$; $k^{\rho\nu}{}_{;\times} = \bar{g}^{\rho\nu}{}_{;\times}$ (derivative with respect to $g_{\mu\nu}$)

4. (a) $h_{\rho\nu;\times} = - \bar{g}_{\mu\nu}\bar{g}_{\rho\sigma}k^{\sigma\mu}{}_{;\times}$; $k^{\rho\nu}{}_{;\times} = - \bar{g}^{\mu\nu}\bar{g}^{\rho\sigma}h_{\sigma\mu;\times}$

   (b) $g^{\sigma\mu}h_{\rho\sigma;\times} + g_{\rho\sigma}k^{\sigma\mu}{}_{;\times} = - h_{\rho\sigma;\times}k^{\sigma\mu} - h_{\rho\sigma}k^{\sigma\mu}{}_{;\times}$

   (c) $\bar{g}^{\sigma\mu}h_{\rho\sigma;\times} = - \bar{g}_{\rho\sigma}k^{\sigma\mu}{}_{;\times}$

5. $\bar{g}^{\sigma\mu}h_{\sigma\mu;\times} = - \bar{g}_{\sigma\mu}k^{\sigma\mu}{}_{;\times} = (\log \frac{\bar{g}}{g})_{,\times}$

6. $h_{\rho\nu;\times\omega} = \bar{g}_{\rho\nu;\times\omega} = - \bar{g}_{\mu\nu}\bar{g}_{\rho\sigma}k^{\sigma\mu}{}_{;\times\omega} - \bar{g}_{\mu\nu}h_{\rho\sigma;\omega}k^{\sigma\mu}{}_{;\times}$

   $- \bar{g}_{\rho\sigma}h_{\mu\nu;\omega}k^{\sigma\mu}{}_{;\times}$

7. $\bar{g}^{\rho\nu}h_{\rho\nu;\times\omega} = - \bar{g}_{\nu\sigma}k^{\sigma\nu}{}_{;\times\omega} - 2\, h_{\rho\sigma;\omega}k^{\sigma\rho}{}_{;\times}$

   $= (\log(\frac{\bar{g}}{g}))_{;\times\omega} - h_{\rho\sigma;\omega}k^{\sigma\rho}{}_{;\times}$

   $= \bar{g}^{\rho\nu}h_{\rho\nu;(\times\omega)}$

8. $h_{\rho\sigma;\omega}k^{\rho\sigma}{}_{;\times} = \bar{g}^{\mu\sigma}\bar{g}^{\rho\nu}h_{\nu\mu;\times}h_{\rho\sigma;\omega}$

B. Derivatives of $\bar{g}_{\mu\nu}$ in terms of $b^{\rho}_{\mu\nu}$

1. $\delta\bar{g}_{\mu\nu;\rho} \equiv \bar{g}_{\mu\nu;\bar{\rho}} - g_{\mu\nu;\rho} = - h_{\mu\nu;\rho} = - 2\, \bar{g}_{\tau(\mu}b^{\tau}_{\nu)\rho}$

2. $h_{\mu[\nu;\rho]} = \bar{g}_{\tau[\nu}b^{\tau}_{\rho]\mu}$

3. $h_{\mu(\nu;\rho)} = \bar{g}_{\tau(\mu}b^{\tau}_{\nu)\rho} + \bar{g}_{\tau(\mu}b^{\tau}_{\rho)\nu} = \bar{g}_{\tau(\nu}b^{\tau}_{\rho)\mu} + \bar{g}_{\tau\mu}b^{\tau}_{\nu\rho}$

4. $h_{(\mu\nu;\rho)} = 2\, \bar{g}_{\tau(\nu}b^{\tau}{}_{\nu\rho)} = \frac{1}{3}(h_{\mu\nu;\rho} + h_{\rho\mu;\nu} + h_{\nu\rho;\mu})$

5. $h_{\mu(\nu;\rho)} - h_{\mu[\nu;\rho]} = h_{\mu\rho;\nu}$

6. $h_{\mu[\nu;\rho]} + h_{\rho[\mu;\nu]} + h_{\nu[\rho;\mu]} = 0 = h_{[\mu\nu;\rho]}$

7. $h_{\mu\nu;\rho\sigma} = \bar{g}_{\mu\nu;\rho\sigma} = 2\, \bar{g}_{\chi(\tau}b^{\tau}_{\mu)\sigma}b^{\tau}_{\nu\rho} + 2\, \bar{g}_{\chi(\tau}b^{\chi}_{\nu)\sigma}b^{\tau}_{\mu\rho}$
$+ 2\, \bar{g}_{\tau(\mu}b^{\tau}_{\nu)\rho;\sigma}$

8. $h_{\mu\nu;[\rho\sigma]} = h_{\tau(\mu}R^{\tau}{}_{\nu)\rho\sigma} = \bar{g}_{\tau(\mu}R^{\tau}{}_{\nu)\rho\sigma} = - \bar{g}_{\tau(\mu}\delta R^{\tau}{}_{\nu)\rho\sigma}$
$= 2\, \bar{g}_{\tau(\mu}b^{\chi}_{\nu)[\rho}b^{\tau}_{\sigma]\chi} + 2\, \bar{g}_{\tau(\mu}b^{\tau}_{\nu)[\rho;\sigma]}$

9. $h_{\mu[\nu;\rho]\sigma} = \bar{g}_{\chi(\tau}b^{\chi}_{\nu)\sigma}b^{\tau}_{\rho\nu} - \bar{g}_{\chi(\tau}b^{\chi}_{\rho)\sigma}b^{\tau}_{\nu\mu} + \bar{g}_{\tau[\nu}b_{\rho]\mu;\sigma}$
$= \bar{g}_{\chi\tau}b_{\sigma}{}^{\chi}{}_{[\nu}b^{\tau}_{\rho]\mu} + \bar{g}_{\chi[\nu}b^{\tau}_{\rho]\mu}b^{\chi}_{\tau\sigma} + \bar{g}_{\tau[\nu}b^{\tau}_{\rho]\mu;\sigma}$

10. $h_{\mu[\nu;|\rho|\sigma]} = 2\, \bar{g}_{\chi(\tau}b^{\chi}_{\mu)[\sigma}b^{\tau}_{\nu]\rho} + \bar{g}_{\chi[\nu}b^{\chi}_{\sigma]\tau}b^{\tau}_{\mu\rho}$
$+ \bar{g}_{\tau[\nu}b^{\tau}{}_{|\mu\rho|;\sigma]} + \bar{g}_{\tau\mu}b_{\rho}{}^{\tau}{}_{[\nu;\sigma]}$

11. $h_{\mu\nu;(\rho\sigma)} = 2\, \bar{g}_{\chi\tau}b^{\chi}_{\nu(\sigma}b^{\tau}_{\rho)\mu} + \bar{g}_{\chi\mu}b^{\chi}_{\tau(\sigma}b^{\tau}_{\rho)\nu} + \bar{g}_{\chi\nu}b^{\chi}_{\tau(\sigma}b^{\tau}_{\rho)\mu}$
$+ 2\, \bar{g}_{\tau(\mu}b^{\tau}_{\nu)(\rho;\sigma)}$

C. $b^{\rho}_{\mu\nu}$ in terms of $h_{\mu\nu}$ and its derivatives.

1. $$b^{\kappa}_{\mu\nu} = \frac{\bar{g}^{\kappa\rho}}{2}(h_{\rho\mu;\nu} + h_{\rho\nu;\mu} - h_{\nu\mu;\rho}) = b^{\kappa}{}_{(\mu\nu)}$$
$$= \frac{\bar{g}^{\kappa\rho}}{2}(h_{\mu[\rho;\nu]} + h_{\nu[\rho;\mu]} + h_{\rho(\mu;\nu)})$$

2. $$b_{\kappa}{}^{\kappa}{}_{\nu} = \frac{\bar{g}^{\kappa\rho}}{2} h_{\rho\kappa;\nu} = \frac{1}{2}(\log \frac{\bar{g}}{g})_{,\nu}$$

3. $$b_{\mu}{}^{\kappa}{}_{\nu;\sigma} = -\frac{1}{2}\,\bar{g}^{\lambda\kappa}\bar{g}^{\omega\rho}h_{\lambda\omega;\sigma}(h_{\rho\mu;\nu} + h_{\nu\rho;\mu} - h_{\nu\mu;\rho})$$
$$+ \frac{\bar{g}^{\kappa\rho}}{2}(h_{\rho\mu;\nu\sigma} + h_{\nu\rho;\mu\sigma} - h_{\nu\mu;\rho\sigma})$$

4. $$b^{\kappa}_{\mu\nu;\kappa} = -\frac{1}{2}\,\bar{g}^{\lambda\kappa}\bar{g}^{\omega\rho}h_{\omega(\lambda;\kappa)}(h_{\rho\mu;\nu} + h_{\rho\nu;\mu} - h_{\nu\mu;\rho})$$
$$+ \frac{\bar{g}^{\kappa\rho}}{2}(h_{\rho\mu;\nu\kappa} + h_{\rho\nu;\mu\kappa} - h_{\nu\mu;\rho\kappa})$$

5. $$b_{\kappa}{}^{\kappa}{}_{\nu;\sigma} = \frac{1}{2}(\log \frac{\bar{g}}{g})_{;\nu\sigma} = b_{\kappa}{}^{\kappa}{}_{(\nu;\sigma)}$$
$$= \frac{1}{2}\,\bar{g}^{\kappa\rho}h_{\rho\kappa;(\nu\sigma)} - \frac{\bar{g}^{\lambda\kappa}}{2}\bar{g}^{\omega\rho}h_{\lambda\omega;\sigma}h_{\rho\kappa;\nu}$$

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13. ABSTRACT

The basic tensors of a Riemannian geometry are found in terms of tensor components by considering the geometry as a field over another arbitrary Riemannian geometry. The approach exhibits symmetries not previously noted. In particular the Riemann tensor of a geometry is found to decompose into a sum of tensors, each with the full symmetry of a Riemann tensor, and each dependent upon only one order of derivative of the metric tensor. Further work to explore the potential value of the approach to general relativity is propcsed.

14 KEY WORDS	LINK A		LINK B		LINK C	
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